OXFORD IB DIPLOMA PROGRAMME

END OF CHAPTER TESTS

MATHEMATICS:
APPLICATIONS AND INTERPRETATION

HIGHER LEVEL

COURSE COMPANION

ENHANCED ONLINE

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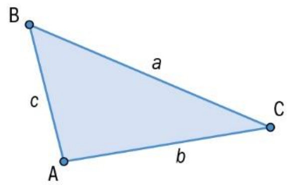
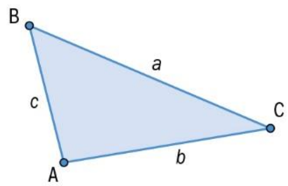
1 Measuring space: accuracy and geometry

1 We consider triangle ABC such that $a\sin(A) = c\sin(C)$.



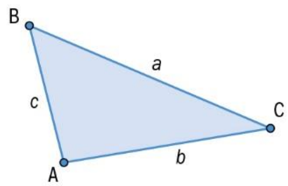
Prove that the above triangle is isosceles.

2 We consider triangle ABC such that $c\cos(A) = a\cos(C)$.



Prove that the above triangle is isosceles.

3 Prove that in every triangle ABC,

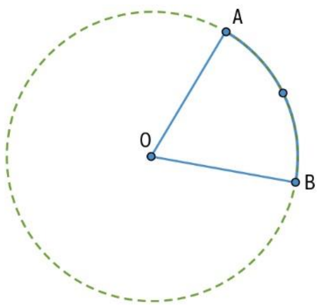


the following equations hold:

a $a = b\cos(C) + c\cos(B)$,

b $\dfrac{\cos(A)}{a} + \dfrac{\cos(B)}{b} + \dfrac{\cos(C)}{c} = \dfrac{a^2 + b^2 + c^2}{2abc}$.

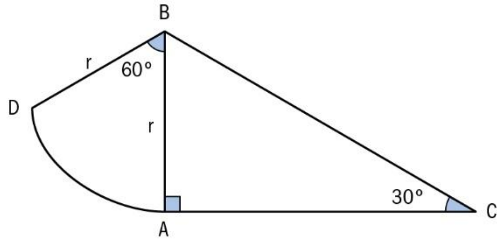
4 We consider a circle which has been shared into 2 sectors in a ratio 5:1 as shown in the non – accurately drawn diagram below.



a Work out the minor angle $A\hat{O}B$ in radians, in terms of π.

b Work out the ratio of the perimeters of the minor sector to the major sector in its simplest form, in terms of π.

5 We consider the following diagram where point B is the centre of radius r of the arc AD, that the angle $B\hat{A}C$ is a right one, that the angle $A\hat{C}B = 30^\circ$ and the arc AD is described by an angle of 60°. All units are measured in cm:



a Find the length of BC in terms of r.

b Find the length of the perimeter of the above diagram, in terms of the radius r.

c Find the total area of the above figure, in terms of r.

d Given that the total perimeter of the shape is $10cm$, find the value of the radius r in $\frac{a}{b+\pi+c\sqrt{d}}$, where a, b, c, d are integers to be found.

Answers

1 From Sine Rule we have: $\frac{\sin(A)}{a}=\frac{\sin(C)}{c}$. But, we know that:

$a\sin(A)=c\sin(C)\Rightarrow\frac{\sin(A)}{c}=\frac{\sin(C)}{a}$. So, by dividing by parts the above equations, we have:

$\frac{c}{a}=\frac{a}{c}\Rightarrow a^2=c^2\Rightarrow a=c$, hence triangle isosceles.

2 From Cosine Rule we have: $\begin{aligned}\cos(C)&=\frac{a^2+b^2-c^2}{2ab}\\ \cos(A)&=\frac{b^2+c^2-a^2}{2bc}\end{aligned}$.

So, we have:

$$\begin{aligned}&a\frac{a^2+b^2-c^2}{2ab}=c\frac{b^2+c^2-a^2}{2bc}\\ &\Rightarrow a^2+b^2-c^2=b^2+c^2-a^2 \quad ,\\ &\Rightarrow 2a^2=2c^2\Rightarrow a=c\end{aligned}$$

hence triangle isosceles.

3 **a** $b\frac{a^2+b^2-c^2}{2ab}+c\frac{a^2+c^2-b^2}{2ac}=\frac{2a^2}{2a}=a$

b $\frac{b^2+c^2-a^2}{2bca}+\frac{a^2+c^2-b^2}{2acb}+\frac{a^2+b^2-c^2}{2abc}=\frac{a^2+b^2+c^2}{2abc}$

4 **a** The circle is shares into 6 parts. Total is 2π radians, hence the minor angle is $\frac{2\pi}{6}=\frac{\pi}{3}$ radians.

b $\dfrac{2\pi r\times\dfrac{\frac{\pi}{3}}{2\pi}+2r}{2\pi r\times\dfrac{2\pi-\frac{\pi}{3}}{2\pi}+2r}=\dfrac{\frac{\pi r}{3}+2r}{\frac{5\pi r}{3}+2r}=\dfrac{\pi+6}{5\pi+6}$

5 **a** $\sin(30^\circ)=\frac{r}{BC}\Rightarrow BC=2r$ cm

b $\tan(30^\circ)=\frac{r}{AC}\Rightarrow AC=r\sqrt{3}$ cm. Length of $AC=2\pi r\frac{60}{360}=\frac{\pi r}{3}$ cm.

So, perimeter $P=r+r+\frac{\pi r}{3}+r\sqrt{3}+2r=3r+\frac{\pi r}{3}+r\sqrt{3}$ cm

c Area $A=\pi r^2\frac{60}{360}+\frac{r\times r\sqrt{3}}{2}=\frac{\pi r^2}{6}+\frac{r^2\sqrt{3}}{2}$ cm^2

$10 = 3r + \frac{\pi r}{3} + r\sqrt{3} \Rightarrow$

d $r = \frac{10}{3 + \frac{\pi}{3} + \sqrt{3}} = \frac{30}{9 + \pi + 3\sqrt{3}}$ cm

2 Representing and describing data: descriptive statistics

Questions 1, 5, 6, 7 can be done by using a GDC.

1 The following table gives the distribution of the time, in seconds, that it took 60 students to cover a specific distance:

t_i	f_i
40	2
45	8
50	10
55	12
60	11
65	10
70	7

a Work out the mean, the median, the mode.

b Work out the standard deviation.

c Work out the time that it took the 25% of the students to cover this distance.

It is given that the standard deviation σ is worked out by using the formula:

$$\sigma^2 = \frac{1}{n}\left(\sum_{i=1}^{k} t_i^2 f_i - \frac{\sum_{i=1}^{k}(t_i f_i)^2}{n}\right)$$, where k is the number of classes.

2 We consider $x_1, \ldots x_n$ observations with mean of $\bar{x}$ and standard deviation of σ_x.

a If $y_1, \ldots y_n$ are observations that we obtain if we add a constant c to each of the $x_1, \ldots x_n$, prove that: $\bar{y} = \bar{x} + c$ and $\sigma_y = \sigma_x$.

b If $y_1, \ldots y_n$ are observations that we obtain if we multiple a constant c to each of the $x_1, \ldots x_n$, prove that:

$\bar{y} = c\bar{x}$ and $\sigma_y = |c|\,\sigma_x$.

3 Eight consecutives even integers have mean of 27. Find these eight numbers. Also, find their median and standard deviation.

4 The mean and the median of 5 numbers are 6. Three of these numbers are 5, 8, 9. Find the other two numbers.

5 The table below shows the number of phone calls that 40 students are receiving in one hour:

number of phone calls	$[0,2)$	$[2,4)$	$[4,6)$	$[6,8)$	$[8,10)$
frequency	6	14	5	10	5

.

Work out:

a the mean

b the modal class

c the median

d the Q_1 and the Q_3

e the standard deviation.

It is given that the standard deviation σ is worked out by using the formula:

$$\sigma^2 = \frac{1}{n}\left(\sum_{i=1}^{k} x_i^2 f_i - \frac{\sum_{i=1}^{k}(x_i f_i)^2}{n}\right)$$, where k is the number of classes.

6 A student bought 10 textbooks that costed, without VAT 20%, £3, 5, 7, 7, 5, 9, 6, 4, 3, 7.

a Find the mean and the standard deviation for the prices of the textbooks without the application of the VAT 20%.

b Find the mean and the standard deviation for the prices of the textbooks with the application of the VAT 20%.

For the standard deviation σ, use the formula: $\sigma^2 = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^{\ 2} - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\right)$

7 In the table below, there is information about the number of people (in thousands) and their age in a town.

We have:

Frequency (in thousands)	11	13	15	10	9
Age Classes (in years)	$[0,15)$	$[15,30)$	$[30,45)$	$[60,75)$	$[75,90)$

.

a Calculate by using interpolation, the low quartile Q_1 and the upper quartile Q_3.

b Calculate the interquartile range (IQR) for the ages of the people in this town.

Answers

1 a $\bar{x} = \dfrac{\sum_{i=1}^{7} t_i f_i}{\sum_{i=1}^{7} f_i} = \dfrac{40\times 2 + 45\times 8 + 50\times 10 + 55\times 12 + 60\times 11 + 65\times 10 + 70\times 7}{60} = \dfrac{170}{3}$ seconds

$\dfrac{60}{2} = 30$, so, to find the median we need the 30th and the 31st value: $\dfrac{55+55}{2} = 55$ seconds

The mode is the time that occurs most frequently: 55 seconds

$\sigma^2 = \dfrac{1}{60}\left(\sum_{i=1}^{7} t_i^2 f_i - \dfrac{(\sum_{i=1}^{7} t_i f_i)^2}{60}\right) =$

b $= \dfrac{1}{60}(40^2\times 2 + \ldots + 70^2\times 7 - \dfrac{1}{60}(40\times 2 + \ldots + 70\times 7)^2) = \dfrac{24{,}119}{9} \Rightarrow$

$\sigma = 51.7676$

c $\dfrac{30}{2} = 15$, so, to find the Q_1 we need the 15th and the 16th value: $\dfrac{50+50}{2} = 50$ seconds

2 a We have that $y_i = x_i + c$. So, $\bar{y} = \dfrac{x_1 + c + \ldots + x_n + c}{n} = \dfrac{x_1 + \ldots + x_n + nc}{n} = \bar{x} + c$

$\sigma_y^2 = \dfrac{(y_1 - \bar{y})^2 + \ldots + (y_n - \bar{y})^2}{n} = \dfrac{(x_1 + c - \bar{x} - c)^2 + \ldots + (x_n + c - \bar{x} - c)^2}{n} = \dfrac{(x_1 - \bar{x}) + \ldots + (x_n - \bar{x})^2}{n} \Rightarrow$

$\sigma_y^2 = \sigma_x^2 \Rightarrow \sigma_y = \sigma_x$

b We have that $y_i = cx_i$. So, $\bar{y} = \dfrac{cx_1 + \ldots + cx_n}{n} = \dfrac{c(x_1 + \ldots + x_n)}{n} = c\bar{x}$

$\sigma_y^2 = \dfrac{(y_1 - \bar{y})^2 + \ldots + (y_n - \bar{y})^2}{n} = \dfrac{(cx_1 - c\bar{x})^2 + \ldots + (cx_n - c\bar{x})^2}{n} = \dfrac{c^2((x_1 - \bar{x}) + \ldots + (x_n - \bar{x})^2)}{n} \Rightarrow$

$\sigma_y^2 = c\sigma_x^2 \Rightarrow \sigma_y = |c|\,\sigma_x$

3 Let $2k,\ 2k+2,\ 2k+4,\ 2k+6,\ 2k+8,\ 2k+10,\ 2k+12,\ 2k+14$ the eight numbers.

Then,

$\dfrac{2k + 2k + 2 + 2k + 4 + 2k + 6 + 2k + 8 + 2k + 10 + 2k + 12 + 2k + 14}{8} = 27 \Rightarrow 2k + 7 = 27 \Rightarrow k = 10$

So, the numbers are: 20, 22, 24, 26, 28, 30, 32, 34.

So, median = $\dfrac{26+28}{2} = 27$.

4 Since the median is 6 and the numbers are five, then one of the two missing numbers must be 6.

Then, if x is the other number, we obtain from the mean: $6 = \dfrac{x + 5 + 6 + 8 + 9}{5} \Rightarrow x = 2$.

So, the missing numbers are 2 and 6.

5 a

number of phone calls	[0,2)	[2,4)	[4,6)	[6,8)	[8,10)
x_i	1	3	5	7	9

$$\bar{x} = \frac{1\times 6 + 3\times 14 + 5\times 5 + 7\times 10 + 9\times 5}{40} = 4.7$$

b $[2,4)$ as it has the highest frequency.

c median $= \frac{20^{\text{st}} + 21^{\text{nd}}}{2} = \frac{3+5}{2} = 4$

d $Q_1 = \frac{9^{\text{th}} + 10^{\text{th}}}{2} = \frac{3+3}{2} = 3$, $Q_3 = \frac{30^{\text{th}} + 31^{\text{st}}}{2} = \frac{7+7}{2} = 7$

e
$$\sigma^2 = \frac{1}{40}\left(\sum_{i=1}^{5} x_i^2 f_i - \frac{\sum_{i=1}^{5}(x_i f_i)^2}{40}\right) =$$
$$= \frac{1}{40}(1^2\times 6 + 3^2\times 14 + 5^2\times 5 + 7^2\times 10 + 9^2\times 5 - \frac{1}{40}(1\times 6 + 3\times 14 + 5\times 5 + 7\times 10 + 9\times 5)^2) \Rightarrow$$
$$\sigma^2 = \frac{671}{100} \Rightarrow \sigma = 2.59$$

6 a $\bar{x} = \frac{3+5+7+7+5+9+6+4+3+7}{10} = £5.80$

$$\sigma^2 = \frac{1}{10}\left(\sum_{i=1}^{10} x_i^{\,2} - \frac{1}{10}\left(\sum_{i=1}^{10} x_i\right)^2\right) =$$
$$= \frac{1}{10}(3^2 + 5^2 + 7^2 + 7^2 + 5^2 + 9^2 + 6^2 + 4^2 + 3^2 + 7^2 - \frac{1}{10}(3+5+7+7+5+9+6+4+3+7)^2)$$
$$= \frac{279}{10} = 27.9 \Rightarrow$$
$$\sigma = £5.28$$

b But, new prices are the old ones' times by 1.2

So, $\bar{x}' = 1.2\bar{x}$
$\sigma' = 1.2\sigma$

So, $\bar{x}' = 5.80 \times 1.2 = £6.96$
$\sigma' = 5.28 \times 1.2 = £6.34$

7 a $11 + 13 + 15 + 10 + 9 = 58$

$Q_1 : \frac{58}{4} = 14.5^{\text{th}}$. So, the low quartile lies in the $[15,30)$ class.

$$\frac{Q_1 - 14.5}{30.5 - 14.5} = \frac{14.5 - (11)}{11 + 13 - 11} \Rightarrow Q_1 = 18.8 \text{ years}$$

$Q_3 : \frac{58\times 3}{4} = 43.5^{\text{th}}$. So, the upper quartile lies in the $[60,75)$ class.

$$\frac{Q_3 - 59.5}{75.5 - 59.5} = \frac{43.5 - (11 + 13 + 15)}{11 + 13 + 15 - (11 + 12 + 15)} \Rightarrow Q_3 = 59.95 \text{ years}$$

b IQR=59.95-18.8=41.15 years .

3 Dividing up space: coordinate geometry, Voronoi diagrams, vectors, lines

Questions 1 and 5 can be done by using a GDC.

1 We consider the vector equations of lines l_1, l_2 :

$$l_1 : \vec{r} = \begin{pmatrix} 7 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}, \lambda \in \mathbb{R}$$
$$l_2 : \vec{r} = \begin{pmatrix} 2 \\ 7 \\ -\frac{3}{2} \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \mu \in \mathbb{R}$$

a Prove that above lines intersect and find the point P of intersection.

b Find the acute angle that the above lines construct to the nearest degree.

We further consider point $A(9, 3, 4)$.

c Verify that the point A lies on the line l_1.

d If point B lies on the line l_2 such that $|AP| = |BP|$, the length $|AB|$.

2 Find the straight line that passes through the point of intersection of the lines
$l_1 : 2x - 5y + 3 = 0$
$l_2 : x - 3y - 7 = 0$

and is perpendicular to the line $l_3 : 4x + y = 1$. Give your answer in $y = mx + c$.

3 Find $k \in \mathbb{R}$ such that $3x + 3y + k = 0$ passes thought the point of intersection of the lines
$$3x + 4y + 6 = 0$$
$$6x + 5y = 9$$

4 Find $\lambda \in \mathbb{R}$ such that the lines $\begin{matrix} (\lambda - 1)x + \lambda y + 8 = 0 \\ \lambda\chi + 3y + 1 - 2\lambda = 0 \end{matrix}$ are perpendicular.

5 Prove that the straight lines $\begin{matrix} x + 4y = 5 \\ 3x - 2y = 1 \\ 7x - 8y = -1 \end{matrix}$, pass though the same point.

6 We consider line $l_1 : 5x + 6y - 7 = 0$ and the line l_2 that crosses x - axis at point $(-4, 0)$ and is perpendicular to l_1.

a Find the equation of the l_2 in the form $y = mx + c$.

The line l_3 is parallel to l_1 and passes through the point $(4, -3)$.

b Find the equation of l_3 in the form $y = mx + c$.

c Find the point of intersection of l_2, l_3.

d Prove that the point of question iii) lies on the curve: $3{,}721(x^2 - y^2) = 244y + 4$.

Answers

1 **a** Lines intersect:

$7+2\lambda=2 \Rightarrow \lambda=\frac{-5}{2}$
$3=7+\mu \Rightarrow \mu=-4$

We verify: $\frac{-3}{2}+3\times(-4)=\frac{-27}{2}=-1+5\times(\frac{-5}{2})$.

Point of intersection: $P(2,3,\frac{-27}{2})$.

$$\begin{pmatrix}2\\0\\5\end{pmatrix}\cdot\begin{pmatrix}0\\1\\3\end{pmatrix}=\left|\begin{pmatrix}2\\0\\5\end{pmatrix}\right|\times\left|\begin{pmatrix}0\\1\\3\end{pmatrix}\right|\times\cos(\theta)$$

b $15=\sqrt{290}\cos(\theta)$
$\cos(\theta)=0.88$

$\theta=28^\circ$, to the nearest degree.

c For $\lambda=1$, the point does lie on the line.

d Finds $|AP|=\sqrt{(9-2)^2+(3-3)^2+(4+\frac{27}{2})^2}=\sqrt{355.25}$

Attempts Cosine Rule: $|AB|=\sqrt{2\times(\sqrt{355.25})^2-2\times(\sqrt{355.25})^2\times0.88}$
$|AB|=\sqrt{85.26}$

$|AB|=9cm$, to the nearest cm.

2 Point of intersection of l_1, l_2: $P(-44,-17)$

Gradient of l_3 : $m=-4$

$$y-(-17)=\frac{1}{4}(x-(-44))$$

So, required line: $y+17=\frac{1}{4}x+11$

$$y=\frac{1}{4}x-6$$

3 Finds the coordinates of the point of intersection: $x=\frac{-2}{3}$, $y=-1$.

Substitutes the point into the first straight line: $3\times(\frac{-2}{3})+4\times(-1)+k=0$
$k=6$

4 Finds both gradients: $\begin{matrix} m_1 = \dfrac{1-\lambda}{\lambda} \\ m_2 = \dfrac{-\lambda}{3} \end{matrix}$.

Since lines are perpendicular: $\dfrac{1-\lambda}{\lambda} \times \dfrac{-\lambda}{3} = -1 \Rightarrow \dfrac{1-\lambda}{-3} = -1 \Rightarrow \lambda = -2$.

5 Finds the point of intersection of any two lines. Here is the point of intersection of the first and the second line: $\begin{matrix} x = 1 \\ y = 1 \end{matrix}$.

Verifies that the found point lies on the third line. Here, verifies that it lies on the third line: $7 \times 1 - 8 \times 1 = -1$.

6 **a** $y = \dfrac{6}{5}(x+4)$

b $y = \dfrac{-5x}{6} + \dfrac{1}{3}$

c $\begin{matrix} x = \dfrac{-134}{61} \\ y = \dfrac{132}{61} \end{matrix}$

d Substitutes the above point to verify that it holds. LHS: $3,721((\dfrac{-134}{61})^2 - (\dfrac{132}{61})^2) = 532$ and RHS: $244 \times \dfrac{132}{61} + 4 = 532$.

4 Modelling constant rates of change: linear functions and regressions

Question 6 can be done by using a GDC.

1 We consider the following functions: $f(x) = \dfrac{ax+b}{x-a}$, with $b \neq -a^2$ and

$g(x) = x - 2\sqrt{x} + 1$ with $0 \leq x \leq 1$.

Prove that:

a $f(f(x)) = x, \forall x \in \mathbb{R} \setminus \{a\}$,

b $g(g(x)) = x, \forall x \in [0,1]$.

2 Find which of the following functions is 1 – 1. For those that are 1 – 1, find the inverse function:

a $f(x) = 2018x + 2019$

b $f(x) = \ln(x-2), x > 2$

3 Given that $f(x) = \sin(x)$ and $g(x) = \sqrt{1+x^2}, x > 0$, find the following functions:

a $f \circ g$

b $g \circ f$

c $g \circ f \circ g$

d $f \circ g^{-1}$

4 **a** We consider the arithmetic progression 1, 3, 5, 7, …

i Prove that the sum of the first n terms is $S_n = n^2$.

ii How many terms do we need to keep adding till we obtain the number 144?

b We consider the arithmetic progression $\{a_i\}$ with common difference of d. Find x:

$3a_7 + (x-3)d = 2a_5 + a_9$.

c If $\dfrac{1}{a+b}, \dfrac{1}{b+c}, \dfrac{1}{a+c}$ are consecutive terms of an arithmetic progression, prove that b^2, a^2, c^2 are also consecutive terms of <u>another</u> arithmetic progression.

5 We have a rope that has length of 20 cm. We are cutting the rope into two ropes, the first has length of x cm and we make a square with this piece. With the rest of the rope we make an equilateral triangle.

Find the function $f(x)$ that expresses the sum of the two areas.

6 In the table below, there is information about the time in minutes, t_i, that a company took to cut down the grass in various lawns of areas a_i:

a_i	330	180	850
t_i	70	40	150

.

a Find the line of regression for the data above.

b Given that this company charges £20 per hour, work out the cost of cutting down the grass from a lawn that has area of 1,100 m^2.

Answers

1 **a** $$f(f(x)) = \frac{a\frac{ax+b}{x-a}+b}{\frac{ax+b}{x-a}-a} = \frac{a(ax+b)+(x-a)b}{ax+b-(x-a)a} = \frac{a^2x+xb}{b+a^2} = \frac{x(a^2+b)}{a^2+b} = x$$

b $$g(g(x)) = x-2\sqrt{x}+1-2\sqrt{x-2\sqrt{x}+1}+1 = x+2-2\sqrt{x}-2\sqrt{(\sqrt{x}-1)^2} =$$
$$= x+2-2\sqrt{x}-2\left|\sqrt{x}-1\right| = x+2-2\sqrt{x}-2(1-\sqrt{x}),\ \text{because } 0 \le x \le 1$$
$$g(g(x)) = x$$

2 **a** $f(a) = f(b) \Rightarrow 2018a+2019 = 2018b+2019 \Rightarrow a = b$, hence f is 1 – 1

$$y = 2018x+2019 \Rightarrow x = \frac{y-2019}{2018} \Rightarrow f^{-1}(x) = \frac{x-2019}{2018}$$

b $f(a) = f(b) \Rightarrow \ln(a-2) = \ln(b-2) \Rightarrow e^{\ln(a-2)} = e^{\ln(b-2)} \Rightarrow a-2 = b-2 \Rightarrow a = b$, hence f is 1 – 1

$$y = \ln(x-2) \Rightarrow e^y = x-2 \Rightarrow x = e^y+2 \Rightarrow f^{-1}(x) = e^x+2$$

3 **a** $f(\sqrt{1+x^2}) = \sin(\sqrt{1+x^2})$

b $g(\sin(x)) = \sqrt{1+\sin^2(x)}$

c $g(\sin(\sqrt{1+x^2})) = \sqrt{1+\sin^2(\sqrt{1+x^2})}$

d $g^{-1}(x) = \sqrt{x^2-1}$

$f(\sqrt{x^2-1}) = \sin(\sqrt{x^2-1})$

4 **a** **i** Arithmetic progression with first term: 1 and common difference: 2

$$S_n = \frac{n}{2}(2\times 1+(n-1)\times 2) = \frac{n}{2}(2n) = n^2$$

ii $144 = n^2 \Rightarrow n = 12$

b
$$3(a_1+6d)+(x-3)d = 2(a_1+4d)+a_1+8d \Rightarrow$$
$$3a_1+18d+(x-3)d = 2a_1+8d+a_1+8d \Rightarrow$$
$$18d+(x-3)d = 16d \Rightarrow$$
$$18+x-3 = 16 \Rightarrow$$
$$x = 1$$

c
$$\frac{1}{b+c} = \frac{\frac{1}{a+c}+\frac{1}{a+b}}{2} \Rightarrow \frac{1}{b+c} = \frac{a+c+a+b}{2(a+c)(a+b)} \Rightarrow 2(a+c)(a+b) = (2a+c+b)(b+c) \Rightarrow$$
$$2a^2+2ab+2ac+2cb = 2ab+2ac+cb+c^2+b^2+bc \Rightarrow$$
$$2a^2 = c^2+b^2 \Rightarrow a^2 = \frac{c^2+b^2}{2}$$

which means that b^2, a^2, c^2 are consecutive terms of an arithmetic progression.

5 Let P_S, P_T be the perimeter of the square and the triangle respectively. We get that:

$$P_S = x$$
$$P_T = 20 - x \; .$$

So, the side of the square will be: $\frac{x}{4}$ and the side of the triangle will be: $\frac{20-x}{3}$

So, if the area of the square is denoted as A_S and the area of the triangle as A_T , we obtain:

$$A_S = (\frac{x}{4})^2 = \frac{x^2}{16} \text{ and } A_T = \frac{1}{2} \times \frac{20-x}{3} \times \frac{20-x}{3} \times \sin(60^\circ) = \frac{\sqrt{3}}{36}(20-x)^2$$

Therefore, $f(x) = \frac{x^2}{16} + \frac{\sqrt{3}}{36}(20-x)^2$.

6 **a** Line of Regression:

$t = A + Ba$

$$B = \frac{S_{at}}{S_{aa}} = \frac{\sum at - \frac{\sum a \times \sum t}{n}}{\sum a^2 - \frac{(\sum a)^2}{n}} =$$

$$= \frac{330 \times 70 + 180 \times 40 + 850 \times 150 - \frac{(330+180+850)\times(70+40+150)}{3}}{330^2 + 180^2 + 850^2 - \frac{(330+180+850)^2}{3}} = 0.1615$$

$$A = \bar{t} - B \times \bar{a} =$$
$$= \frac{70+40+150}{3} - B \times \frac{330+180+850}{3} = 13.4538$$

Hence, $t = A + Ba = 13.4538 + 0.1615a$

b Find the time for 1,100 m^2 : $t = 13.4538 + 0.1615 \times 1100 = 191.1$

5 Quantifying uncertainty: probability

1 In a shop, one can buy something by using either card A or card B. 25% of the costumers have in their possession card A, 55% own card B and 15% own both cards.

a What is the probability that a costumer owns at least one of the two cards?

b What is the probability that a costumer owns only card A and not B?

c What is the probability that a costumer owns card B given that he/she has card A as well?

2 We consider a sample space Ω and two events of it: A, B. It is given that $P(A) = \frac{1}{2}$, $P(B') = \frac{2}{3}$ and $P(A \cap B) = \frac{1}{12}$. Work out the following probabilities:

a $P(A \cup B)$

b $P(B \mid A')$

3 In a box there are 8 red balls and 4 yellow ones. I pick two balls randomly. Work out the probability that both balls that I pick are going to be red.

4 We randomly choose one family with two kids. Work out the probability that the family we chose has two boys given that one of them is a boy.

5 a For two events A, B in a sample space we have that $A, B \subset \Omega$.

i Prove that $P(B) - P(A') \leq P(A \cap B)$.

ii Prove that $2P(A \cap B) \leq P(A) + P(B) \leq 1 + P(A \cup B)$.

b Given further that $P(A) = 0.6$ and that $P(B) = 0.7$,

i Prove that $0.3 \leq P(A \cap B) \leq 0.6$.

ii It is further given for event A that: $\frac{P(A)}{P(A')} = \frac{3}{4}$. Find $P(A')$.

6 In a Sixth Form College there are Year 1, Year 2 classes and a class of GCSE resits that might continue their studies at the College. All together the students are 400. We know that 50 students are in the class of the GCSE resit. We choose one student randomly. Given that the probability that the students is enrolled in Year 2 is 20%, find how many students are enrolled in Year 2 and how many in Year 1.

Answers

1 **a** Let A, B are the events that a costumer owns cars A and card B respectively.

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.25 + 0.55 - 0.15 = 0.65$, so 65%.

b $P(A \cap B') = P(A) - P(A \cup B) = 0.25 - 0.15 = 0.1$, so 10%.

c $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.15}{0.25} = 0.6$, so 60%.

2 **a** $P(B) = 1 - P(B') = 1 - \dfrac{2}{3} = \dfrac{1}{3}$ $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{12} = \dfrac{3}{4}$

b $P(B \mid A') = \dfrac{P(B \cap A')}{P(A')} = \dfrac{P(B) - P(B \cap A)}{1 - P(A)} = \dfrac{\frac{1}{3} - \frac{1}{12}}{1 - \frac{1}{2}} = \dfrac{1}{2}$

3 We consider R_1, R_2 the two events of picking the first ball red and the second one red respectively.

The events R_1, R_2 are independent.

$P(R_1) = \dfrac{8}{12}$ and $P(R_2) = \dfrac{8-1}{12-1} = \dfrac{7}{11}$.

Hence, $P(R_1 \cap R_2) = \dfrac{8}{12} \times \dfrac{7}{11} = \dfrac{14}{33}$.

4 $A = \{\text{boy boy, boy girl, girl boy}\}$, $B = \{\text{both kids are boys}\}$.

$P(A) = \dfrac{3}{4}$ and $P(A \cap B) = \dfrac{1}{4}$.

$P(B \mid A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{\frac{1}{4}}{\frac{3}{4}} = \dfrac{1}{3}$.

5 **a** **i** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0 \le P(A \cup B) \le 1$

$$X = P(B) - (1 - P(A)) - P(A \cap B))$$

$X = P(A) + P(B) - P(A \cap B) - 1 = P(A \cup B) - 1 \le 0$

ii $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

For the LHS inequality: Writes: $P(A \cap B) = P(A) + P(B) - P(A \cup B)$.

Hence:

$$2(P(A) + P(B) - P(A \cup B)) \le P(A) + P(B) \Leftrightarrow 2P(A \cup B) \ge P(A) + P(B)$$

which holds as $\begin{matrix} P(A \cup B) \geq P(A) \\ P(A \cup B) \geq P(B) \end{matrix}$.

<u>For the RHS inequality:</u>

$P(A) + P(B) \leq 1 + P(A) + P(B) - P(A \cap B)$

$P(A \cap B) \leq 1$, which holds.

b i From the inequality: $0 \leq P(A \cup B) \leq 1$, we obtain: $0 \leq P(A) + P(B) - P(A \cap B) \leq 1$

So, $0 \leq 1.3 - P(A \cap B) \leq 1$.

So, $0.3 \leq P(A \cap B) \leq 1.3$, which proves the one side.

However, we know that $A \cap B \subseteq A, B$.

Hence, $P(A \cap B) \leq P(A)$.

So, $P(A \cap B) \leq 0.6$, which proves the second side.

Therefore, $0.3 \leq P(A \cap B) \leq 0.6$.

ii Since, $P(A) + P(A') = 1$, we obtain that: $\dfrac{P(A)}{1 - P(A)} = \dfrac{3}{4}$.

$P(A) = \dfrac{3}{7}$.

6 For Year 1 and Year 2 there are 350 students.

Since the probability of choosing one student to be in Year 2 is 20%, then 350 x 0.2 = 70 students.

Hence, in Year 1 there are 350 − 70 = 280 students.

6 Modelling relationships with functions: power and polynomial functions

All questions can be done by using a GDC.

1 We consider the function $f(x) = \dfrac{x^2 - 3x - 4}{x^2 - 1}$.

a Find the domain of the above function.

b Find the point where the curve of the function crosses the y - axis.

c Is there any point where the above function crosses the x - axis? If yes, find the coordinates of the point(s).

d Examine whether the above function paces through the point $A(-2, 3)$.

e Simplify the function.

2 In a town it is estimated that when the population is x, measured in hundred thousand, there would be $N = 10\sqrt{2(x^2 + x)}$ thousand cars. According to recent reviews that have been heard in this town, in t years from today the population of the town will be $\sqrt{t} + 4$, measured in hundred thousand people.

a Express N as a function of t.

b In how many years from today there will be 120,000 cars in this town?

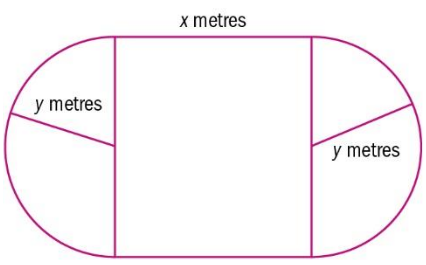
3 Solve the equation: $(x^2 + 2x - 1)^2 - 3(x^2 + 2x + 3) + 14 = 0$.

4 An iceberg is melting down due to global warming. Its estimated volume is given by the formula:

$V(t) = \dfrac{500\pi}{3}(2{,}000 - 100t + 20t^2 + t^3)$, in m^3 after t years from today.

Find after how many years the iceberg is going to ceased to exist due to melting down.

5 The Stadiums in ancient Greece consisted of a rectangle and two semicircles with total perimeter of 400 m. The athletes were performing on this perimeter.



Work out the dimensions of the stadium so that the rectangle has maximum area.

Answers

1 **a** $D_f = \{x \in \mathbb{R} / x \neq \pm 1\}$

b $f(0) = 4$, so: $A(0,4)$

c $x^2 - 3x - 4 = 0 \Rightarrow x = \dfrac{3 \pm \sqrt{9+16}}{2} = \dfrac{3 \pm 5}{2} \Rightarrow x = 4$ or $x = -1$.

So, $B(4,0), C(-1,0)$

d $f(-2) = \dfrac{(-2)^2 - 3(-2) - 4}{(-2)^2 - 1} = \dfrac{6}{3} = 2 \neq 3$, so, the point does not lie on the curve.

e $f(x) = \dfrac{(x-4)(x+1)}{(x+1)(x-1)} = \dfrac{x-4}{x-1}$

2 **a** $N(t) = 10\sqrt{2(\sqrt{t}+4)^2 + 2\sqrt{t} + 8)} = 10\sqrt{2(t + 9\sqrt{t} + 20}$

b $120 = 10\sqrt{2(t+9\sqrt{t}+20)} \Rightarrow 12 = \sqrt{2(t+9\sqrt{t}+20)} \Rightarrow 144 = 2(t+9\sqrt{t}+20) \Rightarrow$
$72 = t + 9\sqrt{t} + 20 \Rightarrow t + 9\sqrt{t} - 52 = 0$

Uses substitution $u = \sqrt{t}$

$$u^2 + 9u - 52 = 0 \Rightarrow u = \frac{-9 \pm \sqrt{81 + 4 \times 52}}{2} = \frac{-9 \pm 17}{2}$$

So, $u = 4$ or $u = -13$ (rejected because $u > 0$)

$u = 4 \Rightarrow \sqrt{t} = 4 \Rightarrow t = 16$ years

3 Let $y = x^2 + 2x - 1$.

$y^2 - 3(y+4) + 14 = 0 \Rightarrow y^2 - 3y + 2 = 0 \Rightarrow y = \dfrac{3 \pm \sqrt{9-8}}{2} = \dfrac{3 \pm 1}{2} \Rightarrow y = 2$ or $y = 1$

$x^2 + 2x - 1 = 2 \Rightarrow x^2 + 2x - 3 = 0 \Rightarrow x = \dfrac{-2 \pm \sqrt{4+12}}{2} = \dfrac{-2 \pm 4}{2} \Rightarrow x = 1$ or $x = -3$

$x^2 + 2x - 1 = 1 \Rightarrow x^2 + 2x - 2 = 0 \Rightarrow x = \dfrac{-2 \pm \sqrt{4+8}}{2} = \dfrac{-2 \pm 2\sqrt{3}}{2} \Rightarrow x = -1 + \sqrt{3}$ or $x = -1 - \sqrt{3}$

4 $V(t) = 0 \Rightarrow 2{,}000 - 100t + 20t^2 - t^3 = 0 \Rightarrow 100(20-t) + t^2(20-t) = 0 \Rightarrow$
$(100 + t^2)(20 - t) = 0 \Rightarrow 100 + t^2 = 0 \Rightarrow$ impossible or t=20

Answer: $t = 20$ years

5 $x + x + 2\pi y = 400 \Rightarrow x + \pi y = 200 \Rightarrow \pi y = 200 - x$

$A = x \times 2y \Rightarrow A(x) = x(200 - x) = 200x - x^2$

We are completing the square to find x corresponding to the maximum value of A:

$A(x) = -(x^2 - 200x) = -((x-100)^2 - 100^2) \Rightarrow A(x) = -(x-100)^2 + 100^2 \Rightarrow x = 100$ m

$\pi y = 200 - 100 \Rightarrow \pi y = 100 \Rightarrow y = \dfrac{100}{\pi}$ m

7 Modelling rates of change: exponential and logarithmic functions

All questions can be done by using a GDC.

1 Find the sum to infinity of a geometric sequence with positive common ratio, if it is given that $a_2 + a_6 = 34$ and $a_3 + a_7 = 68$.

2 Find the first term a_1 and the common ratio $r > 0$ of a geometric sequence if it is given that the sum of the first two terms is $3 + \sqrt{3}$ and that the sum of the first four terms is $4(3 + \sqrt{3})$. Give your answer for both in simplified surd forms.

3 Solve the equation: $(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots)x^3 = 216$.

4 Prove that $\sqrt{a\sqrt{a\sqrt{a\sqrt{a\sqrt{a\cdots}}}}} = a$, $\forall a > 0$.

5 Prove that:

a $\log_2(3) + 2\log_2(4) - \log_2(12) = 2$

b $2\log_2(2 + \sqrt{2}) + \log_2(6 - 4\sqrt{2}) = 2$

6 Show that: $\log_a(\log_a(\sqrt[a]{\sqrt[a]{\sqrt[a]{\cdots\sqrt[a]{a}}}}\,)) = -n$

$\leftarrow n$ roots $\rightarrow$

7 Mary invests £6,500 with annual rate of 8%.

a How much money will she have after 7 years? Give your answer to the nearest pound.

b if the rate is to increase to 9%, how much does she have to invest so that she can have £15,000 after 6 years? Give your answer the nearest pound.

c What must be the annual rate r % for a bank account so that when Mary deposits £150, she can have £250 after 12 years? Give your answer in 2 decimal places.

Answers

1 $a_n = a_1 \times r^{n-1}$, where a_1 and $r \neq 1$ are the first term and the common ration respectively.

$a_2 = a_1 r$
$a_6 = a_1 r^5$
$a_3 = a_1 r^2$
$a_7 = a_1 r^6$

Hence, $\begin{matrix} a_1 r - a_1 r^5 = 7 \\ a_1 r^2 + a_1 r^6 = 112 \end{matrix}$.

By dividing by parts, we obtain: $\dfrac{a_1 r - a_1 r^5}{a_1 r^2 + a_1 r^6} = \dfrac{7}{112} \Rightarrow \dfrac{1 - r^4}{1 + r^4} = \dfrac{1}{16}$.

By solving the above equation, we obtain: $r = \dfrac{1}{2}$.

Substituting to find the a_1: $a_1(\frac{1}{2}) + a_1(\frac{1}{2})^5 = 7 \Rightarrow a_1 = \dfrac{7}{\frac{1}{2} + (\frac{1}{2})^5} = \dfrac{224}{17}$.

Therefore, $S_\infty = a_1 \times \dfrac{1}{1 - r} = \dfrac{224}{17} \times \dfrac{1}{1 - \frac{1}{2}} = \dfrac{448}{17}$.

2 Writes: $\begin{matrix} a_2 = a_1 r \\ a_3 = a_1 r^2 \\ a_4 = a_1 r^3 \end{matrix}$.

So, $\begin{matrix} a_1 + a_2 = 3 + \sqrt{3} \\ a_1 + a_2 + a_3 + a_4 = 4(3 + \sqrt{3}) \end{matrix}$.

So, $\begin{matrix} a_1 + a_1 r = 3 + \sqrt{3} \\ a_1 + a_1 r + a_1 r^2 + a_1 r^3 = 4(3 + \sqrt{3}) \end{matrix}$.

So, $3 + \sqrt{3} + r^2(3 + \sqrt{3}) = 4(3 + \sqrt{3})$.

Solves for r: $r = \sqrt{3}$.

Substitutes to find a_1: $a_1 + a_1\sqrt{3} = 3 + \sqrt{3}$.

So, $a_1 = \sqrt{3}$.

3 Recognises that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ is the sum to infinity of the geometric sequence with fist term: $a_1 = \dfrac{1}{2}$ and common ratio: $r = \dfrac{1}{2}$.

Writes: $S_\infty = \frac{a_1}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$.

Hence: $x^3 = 216 \Rightarrow x = 6$

4 Recognises that the LHS is: $\{a[a(a\cdots)^{\frac{1}{2}}]^{\frac{1}{2}}\}^{\frac{1}{2}} = a^{\frac{1}{2}} \times a^{\frac{1}{2}\times\frac{1}{2}} \times a^{\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}} \times \cdots = a^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots}$.

Writes $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ justifying that it is geometric sequence with $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$.

Hence: RHS is a.

5 **a** $\log_2(3) + \log_2(4^2) - \log_2(12) = \log_2(\frac{3\times 16}{12}) = \log_2(4) = 2$

b $\log_2(2+\sqrt{2})^2 + \log_2(6-4\sqrt{2}) = \log_2((2+\sqrt{2})^2 \times (6-4\sqrt{2})) = \log_2((6+\sqrt{4})\times(6-4\sqrt{2})) =$
$\log_2(4) = 2$

6 For the LHS we have:

$\log_a(\log_a(a^{(\frac{1}{a})^n})) = \log_a((\frac{1}{a})^n \times \log_a(a)) = \log_a((\frac{1}{a})^n \times 1) = n \times \log_a(\frac{1}{a}) = n \times (-1) = -n$

7 **a** $6,500 \times (1+0.08)^7 = £11,140$

b $15,000 = x(1+0.09)^6$

$x = \frac{15,000}{1.09^6} = £8944$

c $250 = 150 \times (1+\frac{r}{100})^{12}$

$(1+\frac{r}{100})^{12} = \frac{250}{150}$

$(1+\frac{r}{100})^{12} = \frac{5}{3}$

$1+\frac{r}{100} = \sqrt[12]{\frac{5}{3}}$

$r = 4.34$

8 Modelling periodic phenomena: trigonometric functions and complex numbers

Questions 1, 2 and 3 can be done by using a GDC.

1 Use your calculator to write each of the following in modulus-argument form:

a $1+2\mathrm{i}$

b $2-3\mathrm{i}$

c $-1-3\mathrm{i}$

d $\sqrt{2}-\sqrt{3}\mathrm{i}$

2 Use your calculator to write each of the following in Cartesian form:

a $3\operatorname{cis}20^\circ$

b $5\operatorname{cis}3$

c $12\mathrm{e}^{\frac{\pi}{12}}$

3 Two electrical sources have their voltages defined as $V_1 = 120\cos\left(30t + \frac{\pi}{6}\right)$ and $V_2 = 180\cos\left(30t + \frac{\pi}{12}\right)$

a When the two voltages are combined, find an expression for the total voltage in the circuit, in the form $a\cos(30t+\alpha)$

b Hence state the total voltage in the system.

4 **a** Plot each of the following on an Argand diagram:

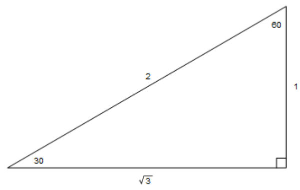
i $1+\sqrt{3}\mathrm{i}$

ii $-\sqrt{3}+\mathrm{i}$

iii $-1-\sqrt{3}\mathrm{i}$

iv $2-2\sqrt{3}\mathrm{i}$

v $-2\sqrt{3}-2\mathrm{i}$



b Use your knowledge of the 30, 60, 90 triangle shown, to write down the modulus and argument of each of the numbers in part **a**

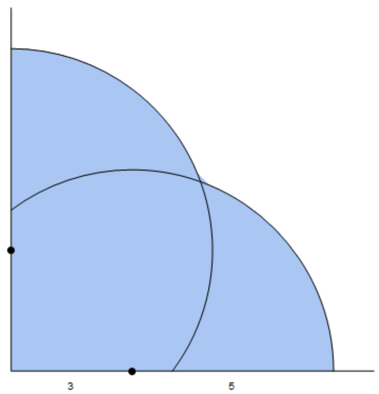
c Hence write each of the numbers in exponential form.

5 Two telephone transmitters are 5km apart and have different transmission strengths. The first has reception within 8 km of the transmitter, and the other within 10 km. Find the area of the region where:

a both transmitters can be received

b just one transmitter can be received

6 In a large room, two security sensors are placed 3 metres either side of a right angled corner. Each sensor can detect movement up to 5 metres away. Find the area of the region where movement can be detected.



7 A man walks on a bearing of 030° for 20km and then on a bearing of 150° for a further 20km. Show that his distance at this point from his starting point can be expressed exactly as $\sqrt{a}$, where *a* is an integer and find the value of *a*.

Answers

1 **a** $2.24\,\text{cis}\,(63.4^\circ)$

b $3.61\,\text{cis}\,(-56.3^\circ)$

c $3.16\,\text{cis}\,(-108^\circ)$

d $2.24\,\text{cis}\,(-50.8^\circ)$

2 **a** $2.82 + 1.03\text{i}$

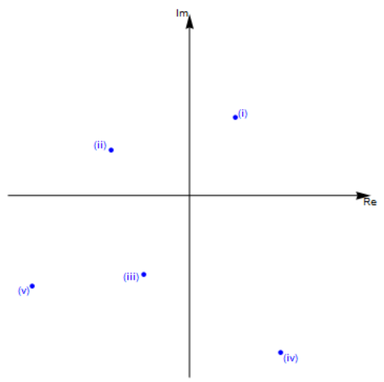
b $-4.95 + 0.706\text{i}$

c $11.6 + 3.11\text{i}$

3 **a** $298\cos\left(30t + 0.366\right)$

b 298V

4 **a**



b **i** $2\,,\ 60^\circ$

ii $2\,,\ 150^\circ$

iii $2\,,\ -120^\circ$

iv $4\,,\ -60^\circ$

v $4\,,\ -150^\circ$

c **i** $2e^{\frac{in}{3}}$

ii $2e^{\frac{5in}{6}}$

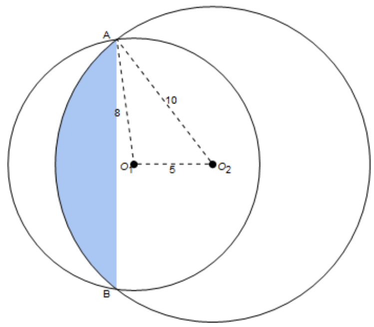
iii $2e^{-\frac{2in}{3}}$

iv $4e^{-\frac{in}{3}}$

v $4e^{-\frac{5in}{6}}$

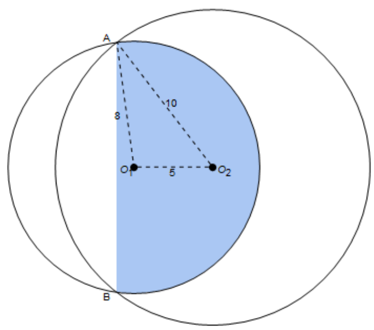
5 **a** Area of left hand side is

$$\frac{1}{2}\times 10^2 \times 2\cos^{-1}\left(\frac{10^2+5^2-8^2}{2\times 10\times 5}\right) - \frac{1}{2}\times 10^2 \times \sin\left(2\cos^{-1}\left(\frac{10^2+5^2-8^2}{2\times 10\times 5}\right)\right) = 43.14$$

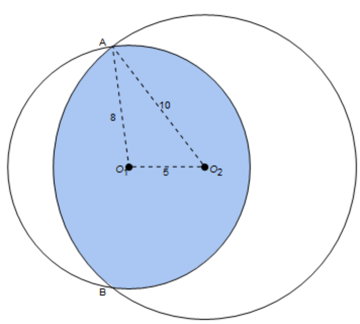


Area of right hand side is

$$\frac{1}{2}\times 8^2 \times 2\cos^{-1}\left(\frac{8^2+5^2-10^2}{2\times 8\times 5}\right)+\frac{1}{2}\times 8^2\times \sin\left(2\pi - 2\cos^{-1}\left(\frac{8^2+5^2-10^2}{2\times 8\times 5}\right)\right)=118.08$$

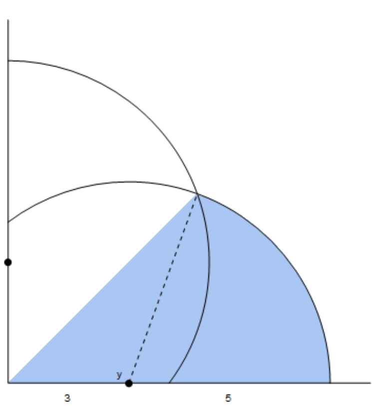


Total area is 161 m^2



b $100\pi + 64\pi - 161\times 2 = 193$ m^2

6



$$y = \pi - \frac{\pi}{4} - \sin^{-1}\left(\frac{3\sin\left(\frac{\pi}{4}\right)}{5}\right)(= 1.918)$$

Shaded area is $\frac{1}{2} \times 5^2 \times (\pi - y) + \frac{1}{2} \times 3 \times 5 \times \sin y = 22.35$

Area covered is therefore 44.7 m^2.

7 $d^2 = 10^2 + 20^2 - 2.10.20\cos 120^\circ = 500 - 400\left(-\frac{1}{2}\right) = 700 \Rightarrow d = \sqrt{700} \Rightarrow a = 700$

9 Modelling with matrices: storing and analysing data

1 Let $M = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ a 2x2 matrix.

a Find the eigenvalues of matrix M.

b Find the corresponding eigenvectors to the eigenvalues of part **a**.

c Hence, write M as PDP^{-1} , where D is a diagonal matrix.

d Express M^n in terms of $n \in \mathbb{Z}$.

e Find the matrix M^{2018}

2 **a** Prove that the identity matrix maps every point (x, y) to itself.

b Find the 2x2 matrix that maps the point (1, 3) to the point (4, 2) and the point (1, 2) to the point (6, 1). Label it as A .

c Find the image of the point (5, 7) under the matrix A .

d If the image of a point is (7, -1), find the point under the transformation of matrix A

3 If $A = \begin{pmatrix} 3 & -1 \\ 2 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & -1 \\ -1 & -7 \end{pmatrix}$, then work out the following matrices:

a $2A - 3B$

b $A^{-1} - B^{-1}$

4 An Operating System could remain in Mode I or could change to Mode II every hour with probabilities 0.6 to change from Mode I to Mode II and probability of 0.6 to change from Mode II into Mode I.

a Construct the transition matrix T and the transition state diagram.

b Using the matrix T , find the probability that the Operating System will change Mode after 2 hours.

c If the System is in Mode I at 3pm find the probability that it will be in Mode I at 6pm the same day.

5 **a** By using matrices, solve the simultaneous equations:

$$3x + 4y = 5$$
$$-x + 6y = 7$$

b In a farm there are some chickens and some pigs. The total amount of legs is 178 and the total amount of heads is 60. Work out the amount of chickens and the amount of pigs that there are in this farm. (Use matrices to solve this problem.)

Answers

1 **a** $\det(\begin{matrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{matrix}) = 0$

$(1-\lambda)(2-\lambda) = 0$

$\lambda = -2$ or $\lambda = 5$

b For $\lambda = -2$,

$$\begin{pmatrix} 1-(-2) & 3 \\ 4 & 2-(-2) \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\bar{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda = 5$,

$$\begin{pmatrix} 1-5 & 3 \\ 4 & 2-5 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\bar{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

c $M = PDP^{-1}$, where $D = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$ and $P = \begin{pmatrix} -1 & 3 \\ 1 & 4 \end{pmatrix}$

d $$M^n = \begin{pmatrix} -1 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} (-2)^n & 0 \\ 0 & 5^n \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & 4 \end{pmatrix}^{-1}$$

$$M^n = \begin{pmatrix} -(-2)^n & 3\times 5^n \\ (-2)^n & 4\times 5^n \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & 4 \end{pmatrix}^{-1}$$

$$M^n = \frac{-1}{7}\begin{pmatrix} -(-2)^n & 3\times 5^n \\ (-2)^n & 4\times 5^n \end{pmatrix}\begin{pmatrix} 4 & -3 \\ -1 & -1 \end{pmatrix} = \frac{-1}{7}\begin{pmatrix} -4\times(-2)^n - 3\times 5^n & 3\times(-2)^n - 3\times 5^n \\ 4\times(-2)^n - 4\times 5^n & -3\times(-2)^n - 4\times 5^n \end{pmatrix}$$

e $$M^{2018} = \frac{-1}{7}\begin{pmatrix} -4\times 2^{2018} - 3\times 5^{2018} & 3\times 2^{2018} - 3\times 5^{2018} \\ 4\times 2^{2018} - 4\times 5^{2018} & -3\times 2^{2018} - 4\times 5^{2018} \end{pmatrix}$$

2 **a** $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

b $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

$a + 3b = 4$
$c + 3d = 2$
$a + 2b = 6$
$c + 2d = 1$

$$A = \begin{pmatrix} 10 & -2 \\ -1 & 1 \end{pmatrix}$$

c Image: $\begin{pmatrix} 10 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 36 \\ 2 \end{pmatrix}$

d $\begin{pmatrix} 10 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 & -2 \\ -1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 7 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5/8 \\ -3/8 \end{pmatrix}$$

3 **a** $\begin{pmatrix} 6 & -2 \\ 4 & 12 \end{pmatrix} + \begin{pmatrix} -15 & 3 \\ 3 & 21 \end{pmatrix} = \begin{pmatrix} -9 & 1 \\ 7 & 33 \end{pmatrix}$

b $\frac{1}{20}\begin{pmatrix} 6 & 1 \\ -2 & 3 \end{pmatrix} - \frac{1}{-36}\begin{pmatrix} -7 & 1 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 19/180 & 1/45 \\ -13/180 & 13/45 \end{pmatrix}$

4 **a** $T = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix}$

$$I \circlearrowleft^{0.4} \underset{0.6}{\overset{0.6}{\rightleftarrows}} II \circlearrowleft^{0.4}$$

b $T^2 = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix}\begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.52 & 0.48 \\ 0.48 & 0.52 \end{pmatrix}$

Answer: 0.48 + 0.48 = 0.96

c Three hours in total.

$$T^3 = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix}\begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix}\begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.496 & 0.504 \\ 0.504 & 0.496 \end{pmatrix}$$

Answer: 0.496

5 **a** $\begin{pmatrix} 3 & 4 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -1 & 6 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{22}\begin{pmatrix} 6 & -4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 1/11 \\ 13/11 \end{pmatrix}$$

b $\begin{aligned} 2x + 4y &= 178 \\ x + y &= 60 \end{aligned}$, where x, y are amount of chickens and pigs respectively.

$$\begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 178 \\ 60 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 178 \\ 60 \end{pmatrix} = \begin{pmatrix} 31 \\ 29 \end{pmatrix}$$

10 Analyzing rates of change: differential calculus

All questions can be done by using a GDC.

1 Show that the tangent $y = x^2$ and $y = \frac{1}{2x} + \frac{1}{2}$ at their common point are perpendicular.

2 Let $f(x) = \frac{ax+a}{x+a}, a \in \mathbb{R}^*$. Find a such that the gradient of the curve f at the point with x - coordinate $x_0 = 0$ is $\frac{1}{2}$.

3 We consider a quadratic curve $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$. Find a, b, c such that the curve passes through point $A(1,2)$ and the line $y = x$ is the tangent to the curve at the origin.

4 Show that the straight line $y = 3x - 2$ has two common points with the curve $y = x^3$. Also, show that the above straight line is tangent to the curve at one of the above common points.

5 If f has second derivative on $\mathbb{R}$ and $f(x^2) = xf(x)$, find the value of $f''(1)$.

6 Show that the curves with equation:

$$C_1 : y = \frac{e^x + e^{-x}}{2}$$
$$C_2 : y = \frac{e^x + e^{-x}}{2}\sin(x), \quad 0 \le x \le \frac{\pi}{2}$$

have the same tangent at their common point.

7 We consider function $f(x) = \sin^2(x), \forall x \in \mathbb{R}$. Prove that, $f''(x) + 4f(x) = 2, \forall x \in \mathbb{R}$.

Answers

1 Finds the point of intersection: $x^2 = \frac{1}{2x} + \frac{1}{2} \Rightarrow x = 1, y = 1$

Finds the tangent line to the $y = x^2$ at point $(1,1)$: $m_1 = \left.\frac{dy}{dx}\right|_{x=1} = 2x|_{x=1} = 2$

Finds the tangent line to the $y = \frac{1}{2x} + \frac{1}{2}$ at the point $(1,1)$: $m_2 = \left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{-1}{2x}\right|_{x=1} = \frac{-1}{2}$

Therefore, because $m_1 \times m_2 = 2 \times (\frac{-1}{2}) = -1$, the tangent lines meet perpendicularly.

2 Finds derivative: $f'(x) = \frac{a(x+a) - (ax+a)}{(x+a)^2} = \frac{a^2 - a}{(x+a)^2}$.

From hypothesis we have: $f'(0) = \frac{1}{2}$.

Obtains: $\frac{1}{2} = \frac{a^2 - a}{(0+a)^2} \Rightarrow a^2 = 2(a^2 - a) \Rightarrow a^2 - 2a = 0 \Rightarrow a(a-2) = 0 \Rightarrow a = 0$ or $a = 2$.

Needs to reject $a = 0$.

Hence, $a = 2$

3 Since $A(1,2)$ lies on the curve then: $f(1) = 2 \Rightarrow 2 = a \times 1^2 + b \times 1 + c \Rightarrow 2 = a + b + c$.

Also, $(0,0)$ lies on the curve. Then: $f(0) = 0 \Rightarrow 0 = a \times 0^2 + b \times 0 + c \Rightarrow c = 0$.

But, $(0,0)$ is the touching point of the tangent line to the curve, which line has gradient 1.

Hence, $f'(0) = 1$. But, $f'(x) = 2ax + b$. So, $1 = 2a \times 0 + b \Rightarrow b = 1$.

Finally, we get: $2 = a + 1 + 0 \Rightarrow a = 1$.

So, $f(x) = x^2 + x$ is the required curve.

4 Attempts to solve the simultaneous equations: $\begin{matrix} y = x^3 \\ y = 3x - 2 \end{matrix} \Rightarrow 3x - 2 = x^3 \Rightarrow x^3 - 3x + 2 = 0$

The latter equation has $x = 1$ as solution.

Attempts long division: $x - 1 \overline{\big) x^3 - 3x + 2}$ with quotient $x^2 + x - 2$

Therefore, $x^3 - 3x + 2 = (x-1)(x^2 + x - 2) = (x-1)(x-1)(x+2) = (x-1)^2(x+2)$.

Hence, $\begin{matrix} x = 1 & \text{or} & x = -2 \\ y = 1 & \text{or} & y = -8 \end{matrix}$.

Therefore, points: $A(1,1), B(-2,-8)$.

Finds tangent lines at both points:

At point $A(1,1)$:
$$\begin{aligned}&y' = 3x^2\\ &m = 3\times 1^2 = 3\\ &y = mx + c \Rightarrow y = 3x + c \Rightarrow 1 = 3\times 1 + c \Rightarrow c = -2\\ &y = 3x - 2\end{aligned}$$

At point $B(-2,-8)$:
$$\begin{aligned}&y' = 3x^2\\ &m = 3\times(-1)^2 = 12\\ &y = mx + c \Rightarrow y = 12x + c \Rightarrow -8 = 12\times(-2) + c \Rightarrow c = 16\\ &y = 12x + 16\end{aligned}$$

Therefore, at point A the straight line is tangent to the curve.

5 Differentiating at both sides once: $f'(x^2)2x = f(x) + xf'(x)$ - Correct use of Product Rule and Chain Rule

Differentiating at both sides again: $f''(x^2)4x^2 + f'(x^2)2 = f'(x) + f'(x) + xf''(x)$ - Correct use of Product Rule and Chain Rule.

Substituting $x = 1$ in the latter equation, we obtain:
$$f''(1^2)\times 4\times 1 + f'(1^2)\times 2 = f'(1) + f'(1) + 1\times f''(1)$$
$$4f''(1) + 2f'(1) = f'(1) + f'(1) + f''(1)$$
$$3f''(1) = 0 \Rightarrow f''(1) = 0$$

6 Finds the common point of the curves: $\dfrac{e^x + e^{-x}}{2} = \dfrac{e^x + e^{-x}}{2}\sin(x) \Rightarrow \dfrac{e^x + e^{-x}}{2}(1 - \sin(x)) = 0$

$\dfrac{e^x + e^{-x}}{2} = 0$ or $\sin(x)=1$

impossible equation $x = \dfrac{\pi}{2}$

Obtains: $x = \dfrac{\pi}{2}$ and $y = \dfrac{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}{2}$

Hence, common point: $A(\dfrac{\pi}{2}, \dfrac{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}{2})$

Finds tangent to C_1 at point $A(\dfrac{\pi}{2}, \dfrac{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}{2})$: $y' = \left.\dfrac{e^x - e^{-x}}{2}\right|_{x=\frac{\pi}{2}} = \dfrac{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}}{2}$

Tangent line: $y - \dfrac{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}{2} = \dfrac{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}}{2}(x - \dfrac{\pi}{2})$

Finds tangent to C_2 at point $A(\dfrac{\pi}{2}, \dfrac{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}{2})$: $y' = \left.\dfrac{e^x - e^{-x}}{2}\sin(x) + \dfrac{e^x + e^{-x}}{2}\cos(x)\right|_{x=\frac{\pi}{2}} = \dfrac{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}}{2}$

Tangent line: $y - \dfrac{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}{2} = \dfrac{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}}{2}(x - \dfrac{\pi}{2})$

Therefore, the curves have the same tangent line at their common point.

7 Finds the first derivative: $f'(x) = 2\sin(x)\cos(x)$.

Finds the second derivative: $f''(x) = 2\cos(x)\cos(x) + 2\sin(x)(-\sin(x)) = 2\cos^2(x) - 2\sin^2(x)$.

LHS: $2\cos^2(x) - 2\sin^2(x) + 4\sin^2(x) = 2\cos^2(x) + 2\sin^2(x) = 2$.

11 Approximating irregular spaces: integration and differential equations

All questions can be done by using a GDC.

1 Work out the following integrals:

a $I_1 = \int_1^2 x^2(x - \frac{1}{x})^2 dx$

b $I_2 = \int_0^4 \frac{x}{\sqrt{x^2+9}} dx$

c $I_3 = \int_0^e \frac{1}{2x\sqrt{\ln(x)}} dx$

d $I_4 = \int_1^2 \frac{3x+2}{x^2+x} dx$

e $I_5 = \int_0^{\frac{\pi}{2}} 3(\sin(x)+2)^{2018} \cos(x) dx$

2 Solve the differential equations:

a $\frac{dy}{dx} = \frac{x^2-1}{y^3}$

b $\frac{dy}{dx} = y(1+\sin(x))$

c $\frac{dy}{dt} = \tan(t)(1+y^2)$

d $x^2dy + 2ydx = 0$

3 We consider the integral $I_n = \int_0^{\frac{\pi}{4}} \tan^n(x) dx$ with $n \geq 3$ and $n \in \mathbb{N}$. Prove that $I_n + I_{n-2} = \frac{1}{n-1}$.

4 We consider function $f(x) = \frac{x^2+x}{x+2}, \forall x \neq -2$.

a Write the above function in the form $ax + b + \frac{c}{x+2}$, where a, b, c are integers to be found.

b Find the $\int_0^1 f(x) dx$.

c Find the $\int_0^1 (2x+1)\ln(x+2) dx$.

5 Prove that $\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{17+\cos(2x)} dx = \frac{\ln(2)}{12}$.

Answers

1 a $I_1 = \int_1^2 x^2(x^2 + \frac{1}{x^2} - 2)dx = \int_1^2 (x^4 + 1 - 2x^2)dx = \frac{x^5}{5} + x - 2 \times \frac{x^3}{3}\Big|_1^2 = \frac{32}{5} + 2 - 2 \times \frac{8}{3} - (\frac{1}{5} + 1 - 2 \times \frac{1}{3}) =$

$= \frac{38}{15}$

b I_2 : use substitution

$u = x^2 + 9 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$, for $x = 0 \Rightarrow u = 9$ and for $x = 4 \Rightarrow u = 25$

$I_2 = \int_9^{25} \frac{1}{2\sqrt{u}} du = \sqrt{u}\Big|_9^{25} = 5 - 3 = 2$

c I_3 : use substitution $u = \ln(x) \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = xdu$, for $x = 1 \Rightarrow u = 0$ and for $x = e \Rightarrow u = 1$

$I_3 = \int_0^1 \frac{1}{2\sqrt{u}} du = \sqrt{u}\Big|_0^1 = 1 - 0 = 1$

d $I_4 = \int_1^2 \frac{2x + 1 + x + 1}{x^2 + x} dx = \int_1^2 \frac{2x + 1}{x^2 + x} dx + \int_1^2 \frac{x + 1}{x^2 + x} dx = \int_1^2 \frac{(x^2 + x)'}{x^2 + x} dx + \int_1^2 \frac{1}{x} dx = \ln(x^2 + x) + \ln(x)\Big|_1^2 =$

$= \ln(12) - \ln(2) = \ln(6)$

e I_5 : use substitution

$u = \sin(x) + 2 \Rightarrow \frac{du}{dx} = \cos(x) \Rightarrow dx = \frac{du}{\cos(x)}$, for $x = 0 \Rightarrow u = 2$ and for $x = \frac{\pi}{2} \Rightarrow u = 3$

$I_5 = \int_2^3 3u^{2018} du = \frac{1}{673}(3^{2019} - 2^{2019})$

2 a $y^3 dy = (x^2 - 1)dx \Rightarrow \int y^3 dy = \int (x^2 - 1)dx \Rightarrow \frac{y^4}{4} = \frac{x^3}{3} - x + c, c \in \mathbb{R}$

b $\frac{1}{y} dy = (1 + \sin(x))dx \Rightarrow \int \frac{1}{y} dy = \int (1 + \sin(x))dx \Rightarrow \ln(y) = x - \cos(x) + c, c \in \mathbb{R}$

c $\frac{1}{1 + y^2} dy = \tan(t)dt \Rightarrow \int \frac{1}{1 + y^2} dy = \int \tan(t)dt \Rightarrow \arctan(y) = -\ln(\cos(t)) + c, c \in \mathbb{R}$

d $\frac{-1}{2y} dy = \frac{1}{x^2} dx \Rightarrow \int \frac{-1}{2y} dy = \int \frac{1}{x^2} dx \Rightarrow \frac{-1}{2} \ln(y) = \frac{-1}{x} + c, c \in \mathbb{R}$

3 $I_n + I_{n-2} = \int_0^{\frac{\pi}{4}} \tan^n(x)dx + \int_0^{\frac{\pi}{4}} \tan^{n-2}(x)dx = \int_0^{\frac{\pi}{4}} \tan^n(x)(1 + \frac{1}{\tan^2(x)})dx =$

$= \int_0^{\frac{\pi}{4}} \tan^n(x) \frac{1}{\sin^2(x)} dx$

Uses substitution $u = \tan(x) \Rightarrow \frac{du}{dx} = \frac{1}{\cos^2(x)} \Rightarrow dx = \cos^2(x)du$

for $x = 0 \Rightarrow u = 0$ and for $x = \frac{\pi}{4} \Rightarrow u = 1$

$$I_n + I_{n-2} = \int_0^1 u^n \frac{1}{\sin^2(x)}\cos^2(x)du = \int_0^1 u^n \frac{1}{\tan^2(x)}du = \int_0^1 u^n \frac{1}{u^2}du = \int_0^1 u^{n-2}du = \left.\frac{u^{n-2+1}}{n-2+1}\right|_0^1 = \left.\frac{u^{n-1}}{n-1}\right|_0^1 =$$

$$= \frac{1}{n-1}$$

4 **a** Long division: $x+2\overline{)x^2+x}$ with quotient $x-1$ with remainder 2

So, $\frac{x^2+x}{x+2} = x-1+\frac{2}{x+2}$

b $\int_0^1 (x-1+\frac{2}{x+2})dx = \left.\frac{x^2}{2} - x + 2\ln(x+2)\right|_0^1 = \frac{-1}{2} + \ln(\frac{9}{4})$

c
$$\int_0^1 (x^2+x)'\ln(x+2)dx = (x^2+x)\ln(x+2)\Big|_0^1 - \int_0^1 (x^2+x)(\ln(x+2))'dx =$$
$$= 2\ln(3) - \int_0^1 \frac{x^2+x}{x+2}dx = 2\ln(3) - (\frac{-1}{2} + \ln(\frac{9}{4})) = \frac{1}{2} + \ln(4)$$

5 $\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{17+1-2\sin^2(x)}dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{9-\sin^2(x)}dx$

Use substitution

$\sin(x) = 3t \Rightarrow \cos(x)dx = 3dt \Rightarrow dx = \frac{3dt}{\cos(x)}$ for $x = 0 \Rightarrow t = 0$ and for $x = \frac{\pi}{2} \Rightarrow t = \frac{1}{3}$

$\frac{1}{2}\int_0^{\frac{1}{3}} \frac{3}{9-9t^2}dt = \frac{1}{6}\int_0^{\frac{1}{3}} \frac{1}{1-t^2}dt = \left.\frac{1}{12}\ln(\frac{1+t}{1-t})\right|_0^{\frac{1}{3}} = \frac{1}{12}\ln(2)$.

12 Modelling motion and change in two and three dimensions

All questions can be done by using a GDC.

1 John and Eve are racing with their cars between points A and B. The position vector of John's car is $r_{eve} = \begin{pmatrix} 4 \\ -7 \end{pmatrix} + t\begin{pmatrix} 10 \\ 11 \end{pmatrix}$, $0 \le t \le 10$. It is given that John's car lies at point A when $t = 0$ sec and at point B when $t = 10$ sec. It is also known that points are given relative to the origin $O(0,0)$ and that distances are measured in metres.

a Find the distance AB to the nearest metre.

b Prove that John keeps a constant speed the whole trip from A to B, which you must find.

Eve's position vector is given by: $r_{eve} = \begin{pmatrix} 4(t^2+2) \\ 5(3t-1) \end{pmatrix}$, $t \in \mathbb{R}^+$.

c How won the race, John or Eve?

d Find the speed of Eve's car at $t = 6$ sec.

e Find the distance between John and Eve when $t = 7$ sec.

2 We consider a particle with mass $m = 3$ kg. On the particle two forces $\bar{F}_1, \bar{F}_2 = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$ are being acted having as a result the change in is motion situation. The position vector is given as $r = \begin{pmatrix} -t^2 \\ 4t^2+1 \end{pmatrix}$, t being the time of its motion.

a Work out the acceleration of the particle.

b Find $\bar{F}_1$.

3 A particle is in equilibrium when on it are acted the following forces: $\bar{F}_1 = \begin{pmatrix} 7 \\ x \\ -4 \end{pmatrix}, \bar{F}_2 = \begin{pmatrix} y \\ 2 \\ 1 \end{pmatrix}, \bar{F}_3 = \begin{pmatrix} 3 \\ 5 \\ z \end{pmatrix}$

a Find the values of x, y, z.

b Given that the particle has mass of 5 kg, find the acceleration of it if we cancel the force $\bar{F}_2$

4 Solve the coupled differential equations $\begin{matrix} y_1' = 3y_2 \\ y_2' = -5y_1 + 4y_2 \end{matrix}$

5 We assume that there are two shops, shop A and shop B, in an area. It is given that the number of costumers choosing shop A than shop B at time t are $x(t)$ and the number of customers choosing B than A at time t are $y(t)$. We also know that for $a, b > 0$, $ax(t)$ customers from shop A decide to go to shop B to shop and $by(t)$ number of costumers decide to go from B to A to shop. Finally, we know that there is a total of 50,000 possible costumers when initially.

a Prove that the above system of differential equations is described by: $\begin{array}{l} \frac{dx}{dt} = -ax + by \\ \frac{dy}{dt} = ax - by \end{array}$.

b Find the number of customers in each shop after many years. (thus, as $t \to \infty$)

Answers

1 a $\overrightarrow{OA} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} 4 \\ -7 \end{pmatrix} + 10\begin{pmatrix} 10 \\ 11 \end{pmatrix} = \begin{pmatrix} 104 \\ 103 \end{pmatrix}$

$AB = \sqrt{(4-104)^2 + (-7-103)^2} = 148.66$ metres

$AB = 149$ metres.

b $v_{john} = \begin{pmatrix} 4+10t \\ -7+11t \end{pmatrix}' = \begin{pmatrix} 10 \\ 11 \end{pmatrix}$ m/s.

$s_{john} = \sqrt{10^2 + 11^2} = \sqrt{221}$ m/s, independent of t, so, constant speed.

c $r_{eve}(10) = \begin{pmatrix} 4(10^2+2) \\ 5(3\times 10 - 1) \end{pmatrix} = \begin{pmatrix} 404 \\ 145 \end{pmatrix}$ m

So, Eve will be $\sqrt{404^2 + 145^2} = 429.233$ m away from point A.

We know that John will cover the 149 m in his 10^{th} sec, so Eve is to be first.

d $v_{Eve} = \begin{pmatrix} 4t^2 + 8 \\ 15t - 5 \end{pmatrix}' = \begin{pmatrix} 8t \\ 15 \end{pmatrix}$ m/s

$s_{eve} = \sqrt{(8\times 6)^2 + 15^2} = \sqrt{2529} = 50.289$ m/s

$s_{eve} = 50$ m/s.

e $d = \left| \begin{pmatrix} 4(7^2+2) \\ 5(3\times 7 - 1) \end{pmatrix} - \begin{pmatrix} 4 \\ -7 \end{pmatrix} - 7\begin{pmatrix} 10 \\ 11 \end{pmatrix} \right| = \left| \begin{pmatrix} 130 \\ 30 \end{pmatrix} \right| = \sqrt{130^2 + 30^2} = \sqrt{17{,}800} = 133.416$ m

$d = 133$ m

2 a $a = \begin{pmatrix} -t^2 \\ 4t^2 + 1 \end{pmatrix}'' \Rightarrow$

$a = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$ m/s^2

b Let $\bar{F}_1 = \begin{pmatrix} x_{\bar{F}_1} \\ y_{\bar{F}_1} \end{pmatrix}$.

Then, $F_{\text{total}} = \begin{pmatrix} x_{F_1} + 0 \\ y_{F_2} + 10 \end{pmatrix} \Rightarrow \begin{pmatrix} x_{F_1} + 0 \\ y_{F_2} + 10 \end{pmatrix} = 3\begin{pmatrix} -2 \\ 8 \end{pmatrix} \Rightarrow \begin{matrix} x_{F_1} = -6 \\ y_{F_1} = 14 \end{matrix}$

$\bar{F}_1 = \begin{pmatrix} -6 \\ 14 \end{pmatrix}$ N

3 **a** $\begin{pmatrix} 7 \\ x \\ -4 \end{pmatrix} + \begin{pmatrix} y \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$7 + y + 3 = 0$
$x + 2 + 5 = 0$
$-4 + 1 + z = 0$

So, $\begin{aligned} x &= -7 \\ y &= -10 \\ z &= 3 \end{aligned}$

$\begin{pmatrix} 7 \\ -7 \\ -4 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix} = 5a \Rightarrow$

b $a = \begin{pmatrix} 2 \\ \frac{-2}{5} \\ \frac{-1}{5} \end{pmatrix}$ m/s^2

4 Finds the eigenvalues: $\begin{vmatrix} 0-\lambda & 3 \\ 5 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 2 \pm \sqrt{19}$

Finds the corresponding eigenvectors:

For $\lambda = 2 + \sqrt{19}$:

$$\begin{pmatrix} -2-\sqrt{19} & 3 \\ -5 & 2-\sqrt{19} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \bar{u}_1 = \begin{pmatrix} 3 \\ 2+\sqrt{19} \end{pmatrix}$$

For $\lambda = 2 - \sqrt{19}$:

$$\begin{pmatrix} -2+\sqrt{19} & 3 \\ 5 & 2+\sqrt{19} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \bar{u}_2 = \begin{pmatrix} 3 \\ 2-\sqrt{19} \end{pmatrix}$$

Hence, $\bar{x}(t) = c_1\begin{pmatrix} 3 \\ 2+\sqrt{9} \end{pmatrix} e^{(2+\sqrt{19})t} + c_2\begin{pmatrix} 3 \\ 2-\sqrt{19} \end{pmatrix} e^{(2-\sqrt{19})t}, c_1, c_2 \in \mathbb{R}$.

5 **a** Since from A to B $ax(t)$ customers go and from B to A $by(t)$ go, then the rate of change of the population of shop A will be $\frac{dx}{dt} = -ax(t) + by(t)$.

Similarly, we obtain that $\frac{dy}{dt} = ax(t) - by(t)$ for the second shop.

b Finds eigenvalues: $\begin{vmatrix} -a-\lambda & b \\ a & -b-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 0$ or $\lambda = -a-b$

Finds corresponding eigenvectors:

For $\lambda = 0$:

$$\begin{pmatrix} -a-0 & b \\ a & -b-0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \bar{u}_1 = \begin{pmatrix} b \\ a \end{pmatrix}$$

For $\lambda = -a-b$:

$$\begin{pmatrix} -a+a+b & b \\ a & -b+a+b \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \bar{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hence, $\bar{x}(t) = c_1\begin{pmatrix} b \\ a \end{pmatrix}e^{0t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-(a+b)t} \Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 b + c_2 e^{-(a+b)t} \\ c_1 a - c_2 e^{-(a+b)t} \end{pmatrix}, c_1, c_2 \in \mathbb{R}$.

$t \to \infty$:

$x(t) \to c_1$b and $y(t) \to c_1 a$, as $e^{-(a+b)t} \to 0$

$t = 0$:

$$\begin{aligned} x(0) &= c_1 b + c_2 \\ y(0) &= c_1 a - c_2 \end{aligned} \Rightarrow x(0) + y(0) = c_1 a + c_1 b$$

But, $x(0) + y(0) = 50{,}000$

So, $c_1 = \dfrac{50{,}000}{a+b}$.

So, Shop A tends to have $\dfrac{50{,}000}{a+b} \times b$ costumers and shop B tends to have $\dfrac{50{,}000}{a+b} \times a$ costumers.

13 Representing multiple outcomes: random variables and probability distributions

All questions can be done by using a GDC.

1 A non-biased die is being thrown 360 times. Find the probability that the sum of the 1200 indications is more or equal to 1200.

2 We consider the random variable X and its probability given by the table below:

i	0	1	2	3	4
$P(X=i)$	$3/7$	$1/7$	$2/7$	$1/14$	$1/14$

.

Work out the EX and the $\text{var}(X)$.

3 We consider the random variable X with cumulative distribution function $f(x)=\begin{cases}\frac{x+1}{3}, -1\le x\le 0\\ \frac{-x+1}{3}, 0\le x\le 1\\ 0, otherwise\end{cases}$.

a Work out $P(X\ge 0.5)$.

b Work out $P(-0.3\le x\le 0.1)$

c Work out the EX and the $\text{var}(X)$.

4 We consider a machine that contracts nails. The length of each nail is a random variable that follows normal distribution (random variable X) with $\mu=6$ cm and $\sigma=0.1$ cm. One nail is considered as "faulty" if its length is more than 6.1 cm or less than 5.9 cm.

a Work out the probability that the machine contracts one "faulty" nail.

b What is the probability that if we pick 20 nails, 1 of them are will be "faulty"?

5 An entomologist studies the number of bugs on the leaves of a tree. We consider that this number follows a Poisson distribution with parameter $\alpha=20$.

Work out the probability that the entomologist,

a picks a leaf with at least 6 bugs on it

b picks 4 leaves and 3 of them have at least 6 bugs on them

Answers

1 Let X the random variable of the indication of the die when it is thrown once. Then $P(X=i)=\frac{1}{6}, i=1,2,3,4,5,6$ as the die is non-biased.

$$EX=\sum_{i=1}^{6} iP(X=i)=1\times\frac{1}{6}+\ldots+6\times\frac{1}{6}=\frac{21}{6}$$

$$\operatorname{var} X=EX^2-(EX)^2=\sum_{i=1}^{6} i^2P(X=i)=1^2\times\frac{1}{6}+\ldots+6^2\times\frac{1}{6}=\frac{91}{6}$$

From the Central Limit Theorem, the sum of the 100 indications (random variable Y) follows

Normal distribution with
$$\mu=n\times EX=360\times\frac{21}{6}=1260$$
$$\sigma^2=n\times\operatorname{var}(X)=360\times\frac{35}{12}=1050$$

$Y\sim N(1260,1050)$

$$P(Y\geq 1200)=P\left(\frac{Y-1260}{\sqrt{1050}}\geq\frac{1200-1260}{\sqrt{1050}}\right)=P\left(Z\geq\frac{-60}{\sqrt{1050}}\right)=P\left(Z\leq\frac{60}{\sqrt{1050}}\right)=F(1.85)=0.9678$$

$$EX=\sum_{i=1}^{5} iP(X=i)=0\times\frac{3}{7}+1\times\frac{1}{7}+2\times\frac{2}{7}+3\times\frac{1}{14}+4\times\frac{1}{14}=\frac{17}{14}$$

2 $\operatorname{var}(X)=EX^2-(EX)^2=\sum_{i=1}^{5} i^2P(X=i)+\left(\sum_{i=1}^{5} iP(X=i)\right)^2=$

$$=0^2\times\frac{3}{7}+1^2\times\frac{1}{7}+2^2\times\frac{2}{7}+3^2\times\frac{1}{14}+4^2\times\frac{1}{14}-\left(\frac{17}{14}\right)^2=\frac{313}{196}$$

3 **a** $P(X\geq 0.5)=\int_{0.5}^{1}\frac{x+1}{3}dx=\left.\frac{x^2}{6}+\frac{x}{3}\right|_{0.5}^{1}=\frac{7}{24}$

b $P(-0.3\leq x\leq 0.1)=\int_{-0.3}^{0.1} f(x)dx=\int_{-0.3}^{0}\frac{x+1}{3}dx+\int_{0}^{0.1}\frac{-x+1}{3}dx=\left.\frac{x^2}{6}+\frac{x}{3}\right|_{-0.3}^{0}+\left.\frac{-x^2}{6}+\frac{x}{3}\right|_{0}^{0.1}=\frac{3}{25}$

c $EX=\int_{-1}^{1} xf(x)dx=\int_{-1}^{0} x\left(\frac{x+1}{3}\right)dx+\int_{0}^{1} x\left(\frac{-x+1}{3}\right)dx=\left.\frac{x^2}{9}+\frac{x^2}{6}\right|_{-1}^{0}+\left.\left(\frac{-x^3}{9}+\frac{x^2}{6}\right)\right|_{0}^{1}=0$

$\operatorname{var}(X)=EX^2-(EX)^2$

$EX^2=\int_{-1}^{1} x^2f(x)dx=\int_{-1}^{0} x^2\left(\frac{x+1}{3}\right)dx+\int_{0}^{1} x^2\left(\frac{-x+1}{3}\right)dx=\frac{2}{9}$

4 **a** $X\sim N(6,0.1^2)$

$Z=\frac{X-6}{0.1}\sim N(0,1)$

$P(X\geq 6.1 \text{ or } X\leq 5.9)=P(X\geq 6.1)+P(X\leq 5.9)=$

$=P\left(Z\geq\frac{6.1-6}{0.1}\right)+P\left(Z\leq\frac{5.9-6}{0.1}\right)=P(Z\geq 1)+P(Z\leq -1)=1-F(1)+F(-1)=$

$=1-F(1)+1-F(1)=2(1-F(1))=2\left(1-\frac{21}{25}\right)=0.32$

b We are to use the Binomial distribution:

$$P(X=1)=\binom{20}{1}\times 0.32^{1}\times(1-0.32)^{20-1}=20\times 0.32\times 0.64^{19}=1.33\times 10^{-3}\ .$$

5 **a** Let X be the random variable that counts insects. $X \sim Po(\alpha = 20)$

$$P(X\geq 6)=1-P(X\leq 5)=1-(e^{-20}\frac{20^{0}}{0!}+e^{-20}\frac{20^{1}}{1!}+e^{-20}\frac{20^{2}}{2!}+e^{-20}\frac{20^{3}}{3!}+e^{-20}\frac{20^{4}}{4!})=0.999$$

b If Y the random variable with 6 bugs on them, then $Y \sim B(4,p)$ with $p=0.999$

$$P(Y=6)=\binom{4}{3}\times 0.999^{3}\times(1-0.999)^{4-3}=0.0004$$

14 Testing for validity: Spearman's hypothesis testing and χ^2 test for independence

All questions can be done by using a GDC.

1 We throw a die 240 times, we note the result each time and he have the following table:

result	1	2	3	4	5	6
frequency	25	46	45	35	55	34

.

Can we claim that this die is not biased? Use 5% significance level.

2 We asked 500 people to write an integer number, with one digit, in a paper (from 0 to 9). We collected all answers which they lie in the following table:

number	0	1	2	3	4	5	6	7	8	9
frequency	44	55	61	49	69	34	44	52	41	51

.

Can we claim that everyone wrote his/her choice of number randomly? Use significance level of 5%

3 The number of accidents that happened in a cross-road the last year are given in the following table:

number of accidents	0	1	2	≥ 3
number of weeks	21	20	7	2

Can we claim that the random variable of number of accidents follows Poisson distribution? Use significant level of 5%.

4 A type of PC is constructed by 5 lines of construction: A, B, C and D in a factory. In a sample of $n = 109$ faulty PCs, we found that 21 were manufactured by line A, 22 by line B, 31 by line C and 35 by line D. Can this sample make us claim that the construction of faulty PCs is the same in all 4 lines? Use significance level of 5%.

5 Let X be a random variable of the weight of 5 people and Y the random variable of the number of books that they have read:

Weight in kg	88	83	101	90	110
Number of books	3	4	5	6	7

.

Calculate the Spearman's Rank Correlation Coefficient between variable X and Y. What type of correlation do we have?

Answers

1 X^2 test :

$H_0 : p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6}$

$H_1 : p_i \neq \frac{1}{6}$, for some i

$\theta_i = n \times p_i = 240 \times \frac{1}{6} = 40$

$$X^2 = \sum_{i=1}^{6}(\frac{n_i^2}{\theta_i}) - n = \frac{25^2}{40} + \frac{46^2}{40} + \frac{45^2}{40} + \frac{35^2}{40} + \frac{55^2}{40} + \frac{34^2}{40} - 240 = 14.3$$

We must reject H_0 when $X^2 > \mathscr{X}^2_{k-1;a}$

But, $\mathscr{X}^2_{6-1;0.05} = 11.07$

So, because: $14.3 > 11.07$, we must reject H_0. So, the die is biased.

2 X^2 test :

$H_0 : p_0 = p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = p_8 = p_9 = \frac{1}{10}$

$H_1 : p_i \neq \frac{1}{10}$, for some $i = 1 = \{0,...9\}$

$\theta_i = p_i \times n = \frac{1}{10} \times 500 = 50$

$$X^2 = \sum_{i=0}^{9}(\frac{n_i^2}{\theta_i}) - n = \frac{44^2}{50} + \frac{55^2}{50} + \frac{61^2}{50} + \frac{49^2}{50} + \frac{69^2}{50} + \frac{34^2}{50} + \frac{44^2}{50} + \frac{52^2}{50} + \frac{41^2}{50} + \frac{51^2}{50} - 500 = 18.44$$

We must reject H_0 when $X^2 > \mathscr{X}^2_{k-1;a}$

$\mathscr{X}^2_{10-1;0.05} = 16.919$

So, because $18.44 < 16.919$, H_0 is accepted, so, each one wrote his/her number of choice randomly.

3 H_0 : random variable follows Poisson distribution
H_1 : reject H_0

$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0,1,2,3$

$p_0 = 0.6065$
$p_1 = 0.3032$
$p_2 = 0.0758$
$p_3 = 0.012$

$\theta_i = np_i$
$\theta_0 = 50 \times 0.6065 = 30.325$
$\theta_1 = 50 \times 0.3032 = 15.16$
$\theta_2 = 50 \times 0.0758 = 3.79$
$\theta_3 = 50 \times 0.012 = 0.6$

$$X^2 = \sum_{i=0}^{3}(\frac{n_i^2}{\theta_i}) - n = \frac{21^2}{30.325} + \frac{20^2}{15.16} + \frac{7^2}{3.79} + \frac{2^2}{0.6} - 50 = 10.52$$

$\mathscr{X}^2_{4-1;0.05} = 7.81$

But, because $10.52 > 7.81$, we must reject the H_0

So, this random variable does not follow Poisson distribution.

4 $H_0 : p_1 = p_2 = p_3 = p_4 = 0.25$
$H_1 : p_i \neq 0.25,$ for some $i = 1, 2, 3, 4$

$\theta_i = n \times p_i$

$\theta = 109 \times 0.25 = 27.25$

$$X^2 = \sum_{i=1}^{4}\frac{(n_i - \theta_i)^2}{\theta_i} = \frac{(21-27.25)^2}{27.25} + \frac{(22-27.25)^2}{27.25} + \frac{(31.27.25)^2}{27.25} + \frac{(35-27.25)^2}{27.25} = 25.86$$

$\mathscr{X}^2_{4-1;0.05} = 7.81473$

So, $X^2 > \mathscr{X}^2_{3;0.05}$, so we reject the H_0

Hence, the construction of the faulty PCs is not the same in all production lines.

5 We make the following table:

x_i	y_i	r_{x_i}	r_{y_i}	d	d^2
88	3	2	1	1	1
83	4	1	2	–1	1
101	5	4	3	1	1
90	6	3	4	–1	1
110	7	5	5	0	0

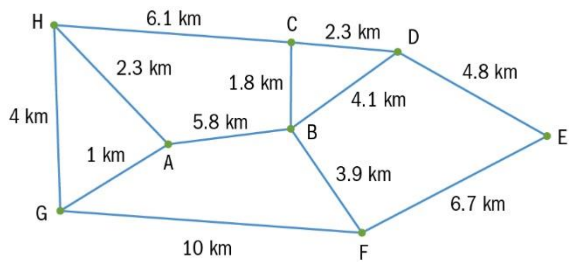
$$r_s = 1 - \frac{6\sum_{i=1}^{5} d_i^2}{5(5^2-1)} = 1 - \frac{6(1+1+1+1+0)}{120} = 0.8$$

Hence, we have positive correlation as the r_s is close to 1.

15 Optimizing complex networks: graph theory

Questions 2 and 3 can be done by using a GDC.

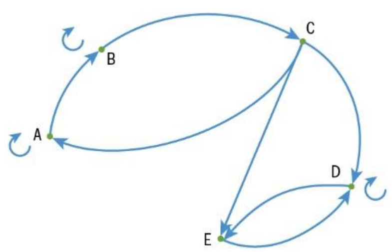
1 The following network shows distances in km between the centres of 8 areas of a bigger city, A to H:



a By using the Prism's algorithm, starting from A, draw the Minimum Spanning Tree diagram.

b Given that the cost of each kilometre costs £670, work out the cost of connecting centres A to H.

2 **a** Find the adjacency matrix M for the following graph:



b Find the number of walks of length 3 from B to C.

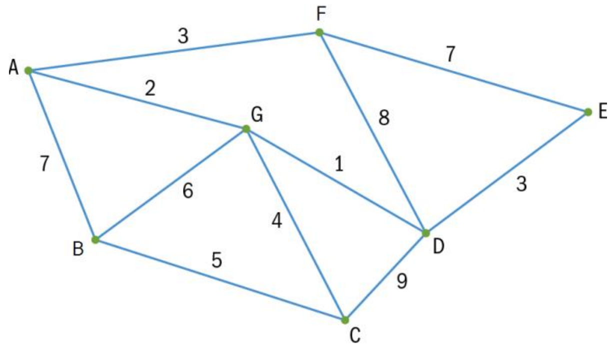
3 An operating System could remain in Mode I or could change to Mode II every hour with probabilities 0.2 to change from Mode I to Mode II and probability of 0.2 to change from Mode II into Mode I.

Construct the transition matrix T and the transition state diagram.

Using the matrix T, find the probability that the Operating System will change Mode after 3 hours.

If the System is in Mode I at 6pm find the probability that it will be in Mode I at 10pm the same day.

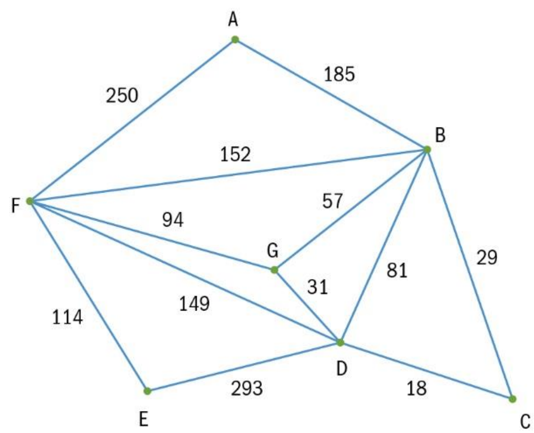
4 On the following network:



a find the Minimum Spanning Tree by using the Kruskal's algorithm

b state the weight of each tree and list the arcs in the order in which you consider them.

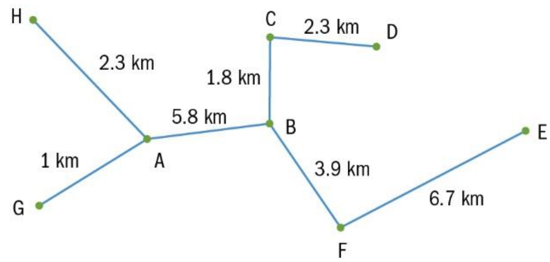
5 We consider the following network:



Starting from vertex A, solve the route inspection problem and find the total length of your route.

Answers

1 **a**



b 2.3 + 1 + 5.8 + 2.3 + 1.8 + 3.9 + 6.7 = 23.8 km

23.8 km x £670 = £15,946

2 **a**

$$M=\begin{pmatrix}1&1&0&0&0\\0&0&1&0&0\\1&0&0&1&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}$$

b Attempts to multiply M^3

Finds: $M^3=\begin{pmatrix}2&2&1&0&0\\2&2&1&1&0\\0&0&1&1&1\\0&0&1&2&1\\0&0&0&1&0\end{pmatrix}$

We have that the required is the entry at the 2nd row and the 3rd column: 0

3 $T=\begin{pmatrix}0.8&0.2\\0.2&0.8\end{pmatrix}$

$$I \circlearrowleft^{0.8} \underset{0.2}{\overset{0.2}{\rightleftarrows}} II \circlearrowleft^{0.8}$$

$$T^3=\begin{pmatrix}0.8&0.2\\0.2&0.8\end{pmatrix}\begin{pmatrix}0.8&0.2\\0.2&0.8\end{pmatrix}\begin{pmatrix}0.8&0.2\\0.2&0.8\end{pmatrix}=\begin{pmatrix}0.68&0.32\\0.32&0.68\end{pmatrix}\begin{pmatrix}0.8&0.2\\0.2&0.8\end{pmatrix}=\begin{pmatrix}0.608&0.392\\0.392&0.608\end{pmatrix}$$

Hence, changing Mode means: $0.392+0.392=0.784$

Four hours in total.

$$T^4 = \begin{pmatrix} 0.608 & 0.392 \\ 0.392 & 0.608 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.5648 & 0.4352 \\ 0.4352 & 0.5648 \end{pmatrix}$$

Hence, the probability that the system will be in Mode I is 0.5648

4 a and b

Arcs:

GD (weight:1)

AG (weight: 2)

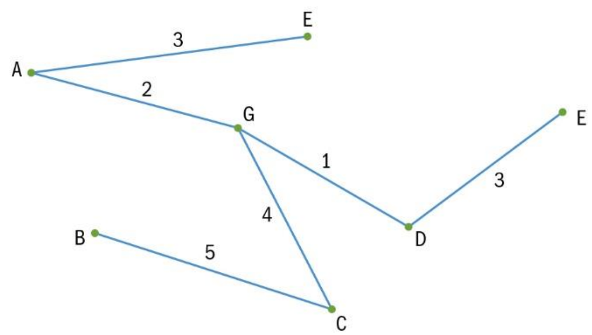
AF, DE (weights:3)

GC (weight: 4)

BC (weight: 5)

Reject: AB, FE, FD, BG, CD

Minimum Spanning Tree:



Weight: 1 + 2 + 3 + 3 + 4 + 5 = 18

5 Odd vertices are: B, G, D, F

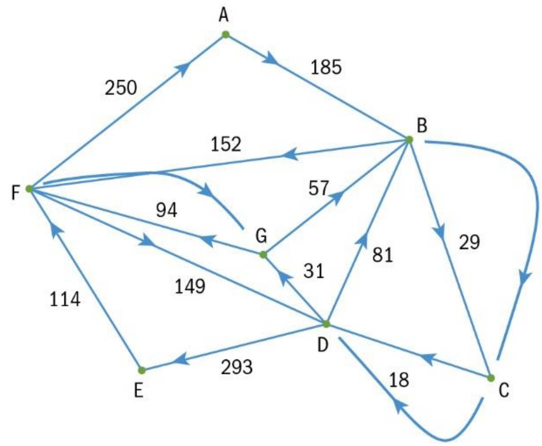
We have:

BG + DF = 57 + 149 = 206

BD + GF = 81 + 94 = 175

BF + GD = 152 + 31 = 183

Repeat arcs BC, CD, BF and add them to the network:



Possible route: A-B-C-D-B-C-D-G-B-F-G-F-D-E-F-A

Weight of this route: 185 + 29 + 18 + 81 + 29 + 18 + 31 + 57 + 152 + 94 + 94 + 149 + 293 + 114 + 250 = 1594